Cohomotopy groups capture robust Properties of Zero Sets via Homotopy Theory

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ACAT meeting 2015
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- In many cases, the preimage of zero (or any single point in $\mathbb{R}^n$) plays a crucial role.
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- Often we have access only to an approximation of the actual map.
Robust features of zero sets

Given $f : X \rightarrow \mathbb{R}^n$, compute features of the zero set $f^{-1}(0)$ that are “stable” with respect to perturbations of $f$. 
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- Interesting cases: \( \dim X \geq n \)
- Stability/robustness is measured by a parameter \( r \in (0, \infty) \) yielding persistence of features
Given $f : X \to \mathbb{R}^n$, compute properties of the zero set $f^{-1}(0)$ that are “stable” with respect to perturbations of $f$. 

$$ f(x, y) = y $$

$R$

$0$
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$$\|g - f\| < r_1$$
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Formalization

For $f : X \rightarrow \mathbb{R}^n$ and $r > 0$, let

$$Z_r(f) := \{g^{-1}(0) : g : X \rightarrow \mathbb{R}^n \text{ s.t. } \|g - f\| < r\}$$

Some robust features of zero sets (properties of $Z_r(f)$) to study:

- The fundamental geometric property of $Z_r(f)$: set of potential zeros
  $\bigcup Z_r(f) = \{x : |f(x)| < r\}$
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- Robust non-emptyness: \( \emptyset \notin Z_r(f) \)
- Robust optima: \( \inf_{Z \in Z_r(f)} \sup_{x \in Z} c(x) \) for some objective \( c : X \to \mathbb{R} \),
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- Robust optima: $\inf_{Z \in Z_r(f)} \sup_{x \in Z} c(x)$ for some objective $c : X \to \mathbb{R}$
- Robust volume: $\inf_{Z \in Z_r(f)} \mathcal{H}^{m-n}(Z)$ where $m = \dim X$
Descriptors of $Z_r(f)$

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**Theorem (A)**

Let $f : X \to \mathbb{R}^n$ and $X$ be compact. If $A_r := \{x : |f(x)| \geq r\}$ is given, then $Z_r(f)$ is determined by the homotopy class of $f/|f| : A_r \to S^{n-1}$. 

$$\text{Descriptors of } Z_r(f)$$
From perturbations to homotopy perturbations

Key idea: perturbations ($g$ with $\|g - f\| < r$) can be replaced by ”homotopy perturbations”:

\[
\text{Lemma } \left\{ g - 1(0) : \|g - f\| < r \right\} = \left\{ e - 1(0) : e|_{\mathcal{A}r} = f|_{\mathcal{A}r} \right\}
\]

Sketch of proof.

$\subseteq (g \Rightarrow e)$: $g|_{\mathcal{A}r} \sim f|_{\mathcal{A}r}$ via straight-line homotopy extends to a homotopy unaffecting the zero set its endpoint is the desired $e$

$\supseteq (e \Rightarrow g)$: multiply $e$ by a scalar function that is 1 of $\mathcal{A}r$ and goes quickly to 0 elsewhere.
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- \(\supseteq\ (e \leadsto g)\):
multiply \(e\) by a scalar function that is 1 of \(A_r\) and goes quickly to 0 elsewhere.
Robust nonemptiness

Immediate consequence:

\[ \emptyset \notin Z_r(f) \iff f/|f|: A_r \to S^{n-1} \text{ can be extended to } X \to S^{n-1} \]

The extendability problem is in decidable when \( \dim X \leq 2n - 3 \) (or \( n = 1, 2 \) or \( n \) even) and is undecidable otherwise.
Theorem (A)

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Descriptors of $Z_r(f)$ continued

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Let $f : X \to \mathbb{R}^n$ and $X$ be compact. If $A_r := \{x : |f(x)| \geq r\}$ is given, then $Z_r(f)$ is determined by the homotopy class of $f/|f| : A_r \to S^{n-1}$.

Moreover, if $A_r \subseteq X$ are CW complexes and $\dim X \leq 2n - 3$, then $Z_r(f)$ is determined by the $\delta$-image of the above homotopy class, where $\delta$ is the “connecting homomorphism” in the sequence

\[ \cdots \to [X, S^{n-1}] \xrightarrow{i^*} [A_r, S^{n-1}] \xrightarrow{\delta} [X/A_r, S^n] \xrightarrow{\cup} [f/|f|] \mapsto [f/A_r] \]
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- The cohomotopy sets are Abelian groups if $\dim X \leq 2n - 4$
- If $\dim X \leq 2n - 4$, the sequence is exact (LES of cohomotopy groups)
  \[\Rightarrow\] Each $[f/A_r]$ uniquely corresponds to the coset $[f/|f|] + i^*[X, S^{n-1}]$ in $[A, S^{n-1}]$. 
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- We denote $\text{Im } \delta$ by $\pi_r$ (group of all descriptors)
- The theorem does not give recipes for how to decode particular robust features from the homotopy class... but it yields a persistence-like tool for distinguishing
When $r$ grows...

$[f/A_r] \in \pi_r$ determines $[f/A_s] \in \pi_s$ for $r < s$ in a structured way, formally:
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- If $r < s$ then the inclusion $A_r \supseteq A_s$ induces $[A_r, S^{n-1}] \rightarrow [A_s, S^{n-1}]$
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$[f/A_r] \in \pi_r$ determines $[f/A_s] \in \pi_s$ for $r < s$ in a structured way, formally:

- If $r < s$ then the inclusion $A_r \supseteq A_s$ induces $[A_r, S^{n-1}] \to [A_s, S^{n-1}]$
- Similarly, there is a map $\pi_r \to \pi_s$ that takes $[f/A_r]$ to $[f/A_s]$
Cohomotopy persistence module

Let $X$ be compact, $\dim X \leq 2n - 3$. Then to each $f: X \to \mathbb{R}^n$ we assign a pointed persistence module $\Pi f$... Im $(\delta) = \pi_r \to \pi_s \to ... \in [f/A_r] \to [f/A_s]$.

Formally, it is a functor from $\mathbb{R}^+$ to the category of pointed Abelian groups (a morphism $(A, a) \to (B, b)$ maps $a$ to $b$).

The assignment $f \mapsto \Pi f$ is stable wrt interleaving distance: $d(\Pi f, \Pi g) \leq \|f - g\|$. After tensoring with a field, $\Pi f$ may be represented via a pointed barcode.
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$$\uplus \quad \uplus$$

$$[f/A_r] \leftrightarrow [f/A_s]$$
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After tensoring with a field, $\Pi_f$ may be represented via a pointed barcode
Theorem (B) Let $X$ be a finite simplicial complex, $f : X \to \mathbb{R}^n$ be simplexwise linear with rational values on vertices and assume $\dim X \leq 2n - 3$. Then the isomorphism type of $\Pi_f$ as well as barcode of $\Pi_f \otimes F$ for $F = \mathbb{Q}$ or $F$ finite can be computed. Main ingredients:

• Computability of cohomotopy groups $[Y, S^{n-1}]$ in the dimension range $\dim Y \leq 2n - 4$. [ˇCadek, K., Matouˇsek, Sergeraert, Vokˇr ´ ınek, Wagner, Computing All Maps into a Sphere]

• Approximation of $A_r$ (up to homotopy equivalence) by simplicial subcomplex $A \Delta_r \subseteq X$.

• Simplicial approximation of $f/|f| : A \Delta_r \to (S^{n-1}) \Delta_r$. 
Computability of $\Pi_f$

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Low dimensional cases

The condition \( \dim X \leq 2n - 3 \) is quite strict for small \( n \), but ...
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- If $n = 1$ (scalar functions) we may easily compute the homotopy class of $f/|f| : A_r \to S^0$ and $[A_r, S^0] \to [A_s, S^0]$ for $r < s$. 

- If $n = 2$, then $f/|f| \in [A_r, S^1]$. This is always a group naturally isomorphic to $H_1(A_r, \mathbb{Z})$.

- If $n = 3$, then $\text{dim } X = 3$ already satisfies $\text{dim } X \leq 2n - 3$.

- However, if $\text{dim } X = 4$ and $n = 3$ then $\emptyset \in \mathbb{Z}^r f$ is undecidable!

- $n = 4$ is nice: $f/|f| \in [A_r, S^3]$ and $[Y, S^3]$ is a group for any $Y$ due to quaternion multiplications – computability of $[Y, S^3]$ is work in progress.
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- If \( n = 2 \), then \( f/|f| \in [A_r, S^1] \). This is always a group naturally isomorphic to \( H^1(A_r, \mathbb{Z}) \).
- \( n = \dim X = 3 \) already satisfies \( \dim X \leq 2n - 3 \). However, if \( \dim X = 4 \) and \( n = 3 \) then \( \emptyset \in Z_r(f) \) is undecidable!
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- **Well groups**: capture homological properties common to all $Z \in Z_r(f)$ (informally)
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  - computability only in special cases: $n = 1$ or $n = \dim X$

- Cap image groups: computable replacement of well groups
  - subgroups of well groups
  - based on primary obstruction to extending $f/|f|$: $A_r \to S^{n-1}$
    - the primary obstruction is the "first component" of $f/A_r$

Our coding effort: compute the secondary (terciary) obstructions and see how much they matter.

Cap image groups can be used to study preimages of all points in $\mathbb{R}^n$ simultaneously in some sense: provide an alternative to multidimensional persistence.
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  - computability only in special cases: $n = 1$ or $n = \dim X$
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  - do not determine $Z_r(f)$

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Cap image groups can be used to study preimages of all points in $\mathbb{R}^n$ simultaneously in some sense: provide an alternative to multidimensional persistence.
Still, the homotopy class \([f_{/A_r}]\) carries more information than needed to encode \(Z_r(f)\). If \(A_r\) is given, then different elements of \(\pi_r\) may determine the same family of zero sets.
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Let \(X\) be a smooth manifold. A function \(g\) is a regular \(r\)-perturbation of \(f\) if \(\|f - g\| < r\) and \(g\) is transverse to \(0 \in \mathbb{R}^n\).
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Z^f_{fr}(f) := \{(g^{-1}(0), dg|_{g^{-1}(0)}): \text{ } g \text{ a regular } r\text{-perturbation of } f\}
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$$Z^\text{fr}_r(f) := \{(g^{-1}(0), dg|_{g^{-1}(0)}) : g \text{ a regular } r\text{-perturbation of } f\}$$

Elements of $Z^\text{fr}_r$ are framed dim $X - n$ dimensional submanifolds of $X$, contained in the complement of $A_r$ (trivialization of the normal bundle).
Optimality of $\Pi_f$

**Theorem**

Assume that $X$ is a smooth compact $m$-manifolds, $r > 0$, $A_r = h^{-1}[0, \infty)$ for some regular $h$, and $m \leq 2n - 3$.

Then there is a bijection

$$\{ Z^\text{fr}_r(f) \mid f : X \to \mathbb{R}^n \text{ such that } A_r = |f|^{-1}[r, \infty) \} \longleftrightarrow \pi_r$$

satisfying that each $Z^\text{fr}_r(f)$ is mapped to $[f/A]$. 

• So, $[f/A]$ is an invariant of $Z^\text{fr}_r(f)$

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\[ Z_{fr}^r(f) \] is a framed cobordism class, then Pontrjagin construction gives the rest.
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- The other direction is more difficult.
Follow the approach of cap image groups. We can construct $\Pi_f(c)$ for any $c \in \mathbb{R}^n$, not just $c = 0$. Can we compute some data structure built from $\Pi_f(c)$, $c \in \mathbb{R}^n$, that robustly describes $f$ itself (not just the zero set)?

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